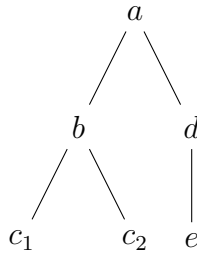


## Homework 2 . Set Theory. Languages.

Scott Weeden - Distance Learner

1. (10) (Tuples). Represent the following tree using a tuple. The goal of this question is to show that tuple is a construct you can use to represent very complex objects.



Tuples are mathematically precise constructs that serve as a *“handy way”* to represent objects with numerous, often recursively defined components. By using ordered nesting, the tuple captures the hierarchical structure of the tree.

The representation starts with the root node ( $a$ ) followed by the tuples representing its children subtrees:

- (a) The root node is  $a$ .
- (b) The first child sub-tree is rooted at  $b$ , with children  $c_1$  and  $c_2$ :  $(b, c_1, c_2)$ .
- (c) The second child sub-tree is rooted at  $d$ , with child  $e$ :  $(d, e)$ .

The formal tuple representation for the entire tree is:

$$(a, (b, c_1, c_2), (d, e))$$

The components of complex systems, like the elements of a Finite Automaton  $((\Sigma, K, s, F, \delta))$ , are themselves represented efficiently using tuples.

2. (5) (Write precise definition). Let  $A = \{1, 2, 3\}$ ,  $B = \{1, 2\}$ ,  $C = \{1, 2, 3\}$ ,  $D = \{1, 2, 3, 4\}$ . Is  $A = B$ ?  $A = C$ ?  $A = D$ ? Write a mathematical definition of = between two sets. (Study how subset definition is written - around P12 of L3. You may directly use concepts such as *subset* defined in the slides.)

Set equality requires two sets to contain **\*\*exactly the same elements\*\***.

- (a) Is  $A = B$ ? **No.**  $A$  contains the element 3, which is not in  $B$ .
- (b) Is  $A = C$ ? **Yes.**  $A$  and  $C$  contain the same elements  $\{1, 2, 3\}$ .
- (c) Is  $A = D$ ? **No.**  $D$  contains the element 4, which is not in  $A$ .

### **Precise Mathematical Definition of Set Equality (=):**

The definition of a concept must use only standard concepts (like logical symbols or membership) or concepts that have been previously defined. Since the query allows using the concept of subset ( $\subseteq$ ), set equality is defined based on mutual subset containment.

**Definition(equal):** For every set  $A$  and  $B$ ,  $A = B$  if and only if  $A \subseteq B$  and  $B \subseteq A$

This definition states that two sets  $A$  and  $B$  are equal if  $A$  is a subset of  $B$  ( $A \subseteq B$ ) and simultaneously  $B$  is a subset of  $A$  ( $B \subseteq A$ ).

Discussion: In mathematical language, defining equality requires leveraging previously defined concepts, in this case, the subset relation ( $\subseteq$ ). The definition relies on the definition of the subset concept itself, which uses the universal quantifier ( $\forall$ ) and implication ( $\rightarrow$ ):  $A \subseteq B$  if and only if for every element  $x$ , if  $x \in A$  then  $x \in B$ .

**Why Mutual Containment?:** Simply checking cardinality ("same number of elements") is insufficient, especially when dealing with infinite sets. The formal definition of  $A = B \iff (A \subseteq B \wedge B \subseteq A)$  is required because it uses established logical and set theoretical definitions to ensure absolute equivalence of elements.

3. (5) (Write precise definition). Let  $A = \{1, 2, 3\}$ ,  $B = \{1, 2\}$ ,  $C = \{a\}$ . What is  $A \cup B$ ?  $A \cup C$ ? Using *set builder notation*, write a precise definition of  $\cup$  between two sets.

The set union operation ( $\cup$ ) is a function over sets. It yields a new set containing all elements present in either operand set.

**Calculations:**

(a)  $A \cup B = \{1, 2, 3\} \cup \{1, 2\} = \{1, 2, 3\}$

(b)  $A \cup C = \{1, 2, 3\} \cup \{a\} = \{1, 2, 3, a\}$

**Precise Definition of Set Union ( $\cup$ ):** A precise definition must rely only on standard concepts (like set, membership ( $\in$ ), and logical connectives). We use set builder notation, which is read as "a set consisting of every  $x$  which makes the statement to be true".

Given the definition of intersection  $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$ , we define union by replacing the conjunction ("and") with the disjunction ("or"):

$$\text{Definition(union): } A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

The resulting set consists of all elements  $x$  such that the condition ( $x \in A$  or  $x \in B$ ) is true.

4. (10) (Concepts and Statements). Consider the statement:

*for any two sets  $A$  and  $B$ ,  $A \cap B \subseteq B \cap A$ .*

- (a) Write each concept in the statement in the format of name with arguments. No need to write logical connectives or quantifiers.
- (b) Draw a tree for the statement  $A \cap B \subseteq B \cap A$ .
- (c) Apply the definition of  $\subseteq$  to its use in statement  $A \cap B \subseteq B \cap A$ . (Recall how a definition of a concept is applied to its use around Page 25 of L2-foundation.)
- (d) If we use the working backward approach to proving the statement  $A \cap B \subseteq B \cap A$ , what (sub-statement) should we prove by working one step backward based on the definition application in (c)?

### Part (a): Concepts and Arguments

A concept is defined mathematically as a function with a name and parameters. Concepts used in the statement, excluding the universal quantifiers ( $\forall A, \forall B$ ) are:

- (a) **Intersection ( $\cap$ )**: This is a standard function over sets.

Intersection( $A, B$ )

- (b) **Intersection ( $\cap$ )**:

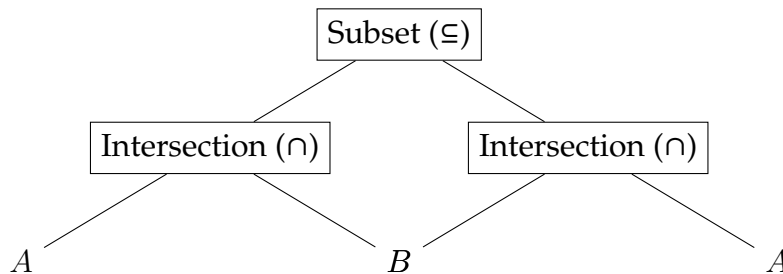
Intersection( $B, A$ )

- (c) **Subset ( $\subseteq$ )**: This is a relation/concept between two sets. The arguments here are the resulting sets from the intersection operations.

Subset( $A \cap B, B \cap A$ )

### Part (b): Structure Tree for $A \cap B \subseteq B \cap A$

The structure tree (or parse tree) illustrates the nested use of concepts. The **\*\*main concept\*\*** (topmost node) governs the entire structure. In this case, the main concept is the subset relation.



### Part (c): Application of the Definition of $\subseteq$

Applying the definition of a concept involves rewriting the general definition by replacing the parameters of the concept with the specific arguments used in the instance across the whole definition.

The precise definition of subset states:

Definition(subset): For every  $A'$  and  $B'$ ,  $A' \subseteq B'$  if for every element  $x$ , if  $x \in A'$  then  $x \in B'$ .

For the statement  $A \cap B \subseteq B \cap A$ :

- Arguments in use:  $A_{arg} = A \cap B$ ,  $B_{arg} = B \cap A$ .
- Parameters in definition:  $A'$ ,  $B'$ .

Replacing the parameters with the arguments yields the instance of the definition:

$A \cap B \subseteq B \cap A$  if for every element  $x$ , if  $x \in (A \cap B)$  then  $x \in (B \cap A)$

### Part (d): Sub-statement to Prove (Working Backward)

The **working backward** methodology dictates that we follow the structure tree from top to bottom, applying logical rules at each step to determine the prerequisite statement that needs to be proven.

The result from (c) shows that proving  $A \cap B \subseteq B \cap A$  (let this be statement  $P$ ) is equivalent to proving the consequence of the definition (let this be statement  $Q$ ). Since the definition statement is assumed to be true, proving  $P$  requires proving  $Q$ .

Working backward one step from the relation  $A \cap B \subseteq B \cap A$  based on the application of the subset definition, we must prove the core quantified implication:

For every element  $x$ , if  $x \in (A \cap B)$  then  $x \in (B \cap A)$

(Or using symbols:  $\forall x, (x \in (A \cap B) \implies x \in (B \cap A))$ ). This decomposition transforms the problem from proving a set relation to proving a statement about arbitrary element membership.

5. (10) (Concepts and Statements). For the definition of *functions* (around P28 in L3), answer the following questions (refer to pages around 12 and 13 in the slides for the answer of some similar question):

- (a) Write each concept, in the format of name and parameters, that is defined in that definition.
- (b) Write each concept, in the format of name and arguments, that is used to define the concepts being defined in the definition.
- (c) Write logical symbols (connectives and quantifiers) used in the definition.
- (d) Write variables used in the definition.

The concept of a function is defined as a specialized type of relation, requiring a comprehensive analysis of the concepts used and logical symbols required to establish its rigor.

[i.]

- (a) **Concepts defined (Name and Parameters):** The primary concept being defined is the property that a given relation constitutes a function from one set to another. We present the concept name followed by its parameters:

$$\mathbf{function}(f, A, B)$$

where  $f$  is the relation being defined,  $A$  is the domain set, and  $B$  is the co-domain set. The use of bold font in the source material indicates the concept being defined.

- (b) **Concepts used to define the concepts being defined (Name and Arguments):** The definition relies on fundamental concepts established earlier in set theory:

- **Relation:** The definition begins by asserting that  $f$  must first be a relation on sets  $A$  and  $B$ .

$$relation(f, A, B)$$

- **Membership ( $\in$ ):** Used to express that elements belong to sets or that a tuple belongs to the relation.

$$membership(a, A)$$

$$membership((a, b), f)$$

- **Tuple:** The relation  $f$  consists of ordered pairs  $(a, b)$ , requiring the concept of a tuple.

$$tuple(a, b)$$

- **Unique Value:** Although often implied by logical structure, the crucial requirement that the output  $b$  must be unique for every input  $a$  is a core concept used.

$$unique(b)$$

- (c) **Logical symbols (connectives and quantifiers) used in the definition:** The constraints defining a function are expressed using standard logical symbols:

- **Universal Quantifier ( $\forall$ ):** Used in the phrase "for every  $a \in A$ ".
- **Existential Quantifier ( $\exists$ ):** Used in the phrase "there is a unique value  $b$  of  $B$ ".
- **Implication ( $\rightarrow$  or *if/then*):** Implicit in the structure "A relation  $f$  is a function... *\*\*if\*\** [conditions apply]".

(d) **Variables used in the definition:** Variables are names used to represent general objects (or parameters/arguments) within the definition statement. The variables used here are:

**$f, A, B, a, b$**

where  $f$ ,  $A$ , and  $B$  represent the defining sets/relation, and  $a$  and  $b$  represent arbitrary elements from those sets subject to the function's constraints.