CS 5384-001/-D01 Logic for Computer Scientists
Homework 1 Solutions

Graduate Student

Due: October 3, 2025

Problem 1

Problem Statement: Prove or disprove if $(A \to (B \lor C)) \leftrightarrow ((A \land C) \lor (A \lor B))$ is a tautology. Use tableau tree and confirm your result with a truth table.

Solution

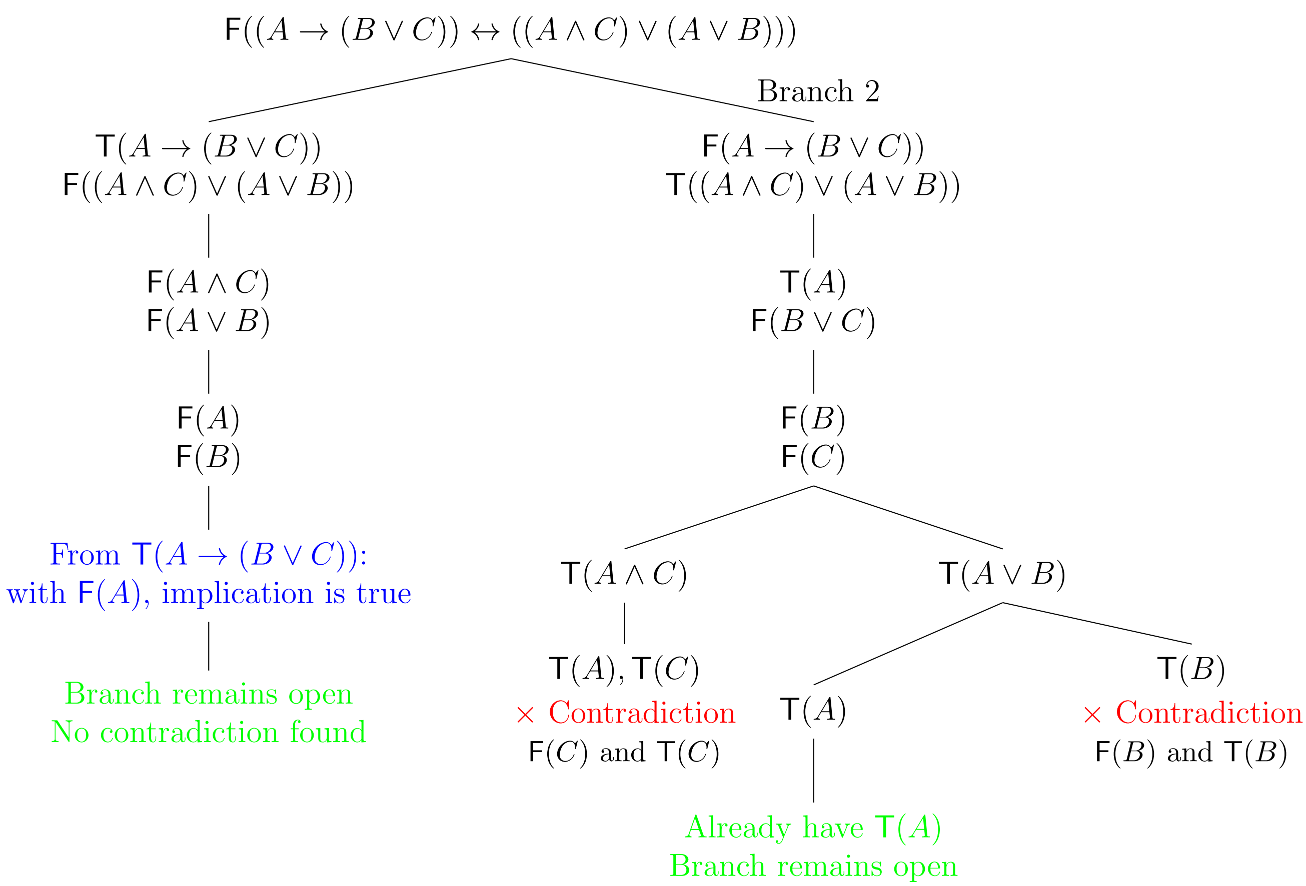
Construct tableau tree to test whether the formula is a tautology, then verify the result using a truth table.

Part 1: Tableau Tree Analysis

To prove whether the formula is a tautology, assume its negation and attempt to derive a contradiction. If all branches close, the formula is a tautology.

Let $\varphi = (A \to (B \lor C)) \leftrightarrow ((A \land C) \lor (A \lor B))$

Start with $\mathsf{F}(\varphi)$ and build the tableau systematically. Since this is a biconditional that is false, one side must be true and the other false.



The tableau shows that not all branches close and that Branch 1 and Case 2a of Branch 2 remain open. Therefore, the formula is not a tautology.

Part 2: Truth Table Verification

Construct truth table to verify this result:

A	B	C	$B \lor C$	$A \to (B \lor C)$	$A \land C$	$A \lor B$	$(A \land C) \lor (A \lor B)$	$\leftrightarrow$
0	0	0	0	1	0	0	0	**0**
0	0	1	1	1	0	0	0	**0**
0	1	0	1	1	0	1	1	1
0	1	1	1	1	0	1	1	1
1	0	0	0	0	0	1	1	**0**
1	0	1	1	1	1	1	1	1
1	1	0	1	1	0	1	1	1
1	1	1	1	1	1	1	1	1

Conclusion: The formula is not a tautology. The truth table shows it evaluates to false in three cases (rows 1, 2, and 5), confirming our tableau analysis. The counterexamples occur when:

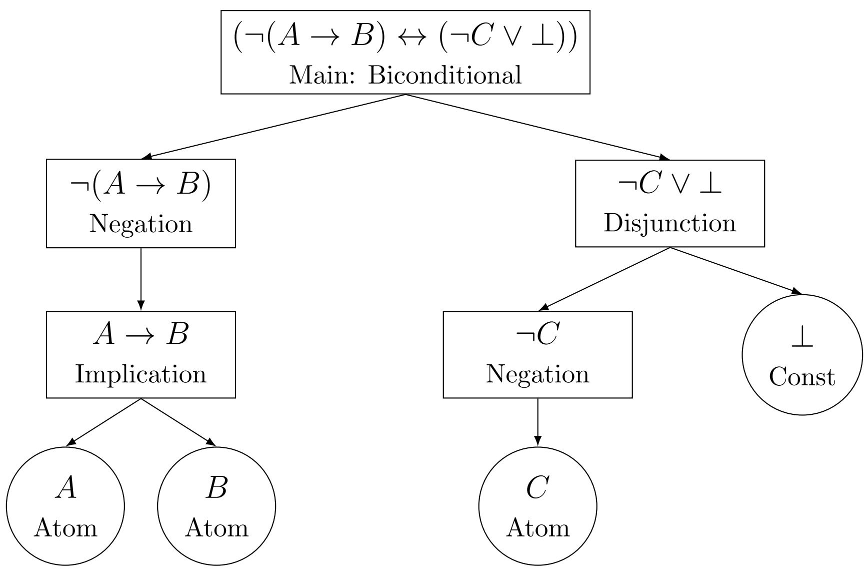
- $A = F, B = F, C = F$ (row 1)
- $A = F, B = F, C = T$ (row 2)
- $A = T, B = F, C = F$ (row 5)

Problem 2

Problem Statement: Using backward method to prove $(\neg(A \to B) \leftrightarrow (\neg C \vee \bot))$ is a proposition.

Solution

The backward method involves decomposing the formula step by step to verify it is well-formed. Construct a parse tree that shows the hierarchical structure of the formula, working from the main connective down to the atomic propositions.



Working backward through the parse tree, I can verify that each component is well-formed:

1. **Main Level:** The root is a biconditional ($\leftrightarrow$) connecting two well-formed formulas.
2. **Left Branch:** $\neg(A \to B)$
   - This is a negation operator applied to an implication
   - The implication $A \to B$ connects two atomic propositions
   - Both A and B are well-formed atomic propositions
   - Therefore, $\neg(A \to B)$ is well-formed
3. **Right Branch:** $(\neg C \vee \bot)$

- This is a disjunction between two formulas
- $\neg C$ is the negation of atomic proposition C, which is well-formed
- $\bot$ is a well-formed logical constant (representing falsehood)
- Therefore, $(\neg C \vee \bot)$ is well-formed

Since both operands of the biconditional are well-formed and the biconditional is a valid binary connective we have a valid proposition.

Simplifying:

$$(\neg C \vee \bot) \equiv \neg C \quad \text{(Identity law: } X \vee \bot \equiv X\text{)} \tag{1}$$
$$\neg(A \to B) \equiv \neg(\neg A \vee B) \equiv A \wedge \neg B \quad \text{(Negation of implication)} \tag{2}$$
$$\text{Therefore: } (A \wedge \neg B) \leftrightarrow \neg C \tag{3}$$

$(A \wedge \neg B) \leftrightarrow \neg C$ clearly shows the formula is a well-formed proposition relating three atomic propositions A, B, and C.

Problem 3

Problem Statement: Begin with $F = \neg((A \to C) \wedge \neg B)$. An equivalent formula for F is $X \to Y$. Express X, Y in terms of A, B, C.

Solution

Transform $F = \neg((A \to C) \wedge \neg B)$ into implication form $X \to Y$ through a series of logical equivalences.

Step 1: Apply De Morgan's Law

First, distribute the negation over the conjunction:

$$F = \neg((A \to C) \wedge \neg B) \tag{4}$$
$$\equiv \neg(A \to C) \vee \neg\neg B \quad \text{(De Morgan's Law)} \tag{5}$$
$$\equiv \neg(A \to C) \vee B \quad \text{(Double negation elimination)} \tag{6}$$

Step 2: Expand the Negation of Implication

Simplify $\neg(A \to C)$:

$$\neg(A \to C) \equiv \neg(\neg A \vee C) \quad \text{(Definition of implication)} \tag{7}$$
$$\equiv \neg\neg A \wedge \neg C \quad \text{(De Morgan's Law)} \tag{8}$$
$$\equiv A \wedge \neg C \quad \text{(Double negation elimination)} \tag{9}$$

Step 3: Substitute Back

Replacing the simplified expression:

$$F \equiv (A \wedge \neg C) \vee B \tag{10}$$

Step 4: Convert to Implication Form
To express this as an implication, use the logical equivalence $P \lor Q \equiv \neg Q \to P$:

$$(A \land \neg C) \lor B \equiv \neg B \to (A \land \neg C) \tag{11}$$

Therefore, the answer is:

$$\boxed{X = \neg B, \quad Y = A \land \neg C}$$

Verification
Verify this result by converting the implication back to disjunction:

$$\begin{aligned} \neg B \to (A \land \neg C) &\equiv \neg\neg B \lor (A \land \neg C) && (12) \\ &\equiv B \lor (A \land \neg C) && (13) \\ &\equiv (A \land \neg C) \lor B \quad \checkmark && (14) \end{aligned}$$

This matches our simplified form of F, confirming our answer is correct.

Problem 4

Problem Statement: Analyze the valuation of the functions:

1. $(A \to B) \leftrightarrow (A \lor \neg B)$
2. $((A \to B) \leftrightarrow (\neg C \lor D))$

Solution

Part 1: $(A \to B) \leftrightarrow (A \lor \neg B)$

Analyze the relationship between the two sides of this biconditional by constructing a truth table:

A	B	$\neg B$	$A \to B$	$A \lor \neg B$	Biconditional	Analysis
0	0	1	1	1	**1**	Both sides true
0	1	0	1	0	**0**	Sides differ
1	0	1	0	1	**0**	Sides differ
1	1	0	1	1	**1**	Both sides true

Analysis:

- The formula is **satisfiable** (true in some cases) but not a tautology
- It evaluates to **true** in exactly 2 out of 4 cases (50%)
- Pattern: The formula is true when A and B have the same truth value
- This formula essentially tests whether $A \equiv B$ (logical equivalence of A and B)
- The formula is contingent - its truth depends on the specific valuation

Part 2: $(A \to B) \leftrightarrow (\neg C \lor D)$

This formula involves four propositional variables. Evaluate truth conditions

A	B	C	D	$A \to B$	$\neg C$	$\neg C \lor D$	Biconditional
0	0	0	0	1	1	1	1
0	0	0	1	1	1	1	1
0	0	1	0	1	0	0	0
0	0	1	1	1	0	1	1
0	1	0	0	1	1	1	1
0	1	0	1	1	1	1	1
0	1	1	0	1	0	0	0
0	1	1	1	1	0	1	1
1	0	0	0	0	1	1	0
1	0	0	1	0	1	1	0
1	0	1	0	0	0	0	1
1	0	1	1	0	0	1	0
1	1	0	0	1	1	1	1
1	1	0	1	1	1	1	1
1	1	1	0	1	0	0	0
1	1	1	1	1	0	1	1

Analysis:

- The formula is **contingent** (neither tautology nor contradiction)
- It evaluates to true in exactly 9 out of 16 cases (56.25%)
- The formula is true when $(A \to B)$ and $(\neg C \lor D)$ have the same truth value
- Note that $(\neg C \lor D) \equiv (C \to D)$, so this formula tests when two implications have equal truth values
- The left side depends only on A and B, while the right side depends only on C and D
- This creates an interesting independence pattern where the formula's truth value depends on whether two separate implications happen to align